definite symmetric matrices, they are all different and positive. Thus the λ_i are the pertinent "relaxation" times of the process. If (l_r/V_r) is a reference macroscopic characteristic time, then the conditions of statement 1 amount to the inequalities $(\lambda_i l_r/V_r) \ll 1$ for any i between 1 and n. Consequently, the time rate of change of $\mathbf{B}_1$ can be neglected with respect to $\mathbf{B}_1$ in Eq. [4], which, accounting the second of Eqs. [3], will accordingly yield

$$\mathbf{A}_1 = \mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) = v_0(\mathbf{L}^{-1}\cdot\boldsymbol{\varphi})\nabla\cdot\mathbf{V}_1$$

Substituting this value of $\mathbf{A}_1$ into Eq. [2a] results in

$$p_1 = -(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi})v_0\nabla\cdot\mathbf{V}_1 - (a_{e0}^2/v_0^2)v_1$$

The positive definite character of the matrix $\mathbf{L}$ implies that $(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}) > 0$. One can then define an essentially positive kinematic volume viscosity ν_v as

$$\nu_v = v_0(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}) \qquad [5]$$

and the basic system becomes, simply

$$\begin{aligned} Dv_1/Dt &= v_0\nabla\cdot\mathbf{V}_1 \\ D\mathbf{V}_1/Dt &= v_0\nabla[\nu_v\nabla\cdot\mathbf{V}_1 + (a_{e0}^2/v_0^2)v_1] \end{aligned}$$

thus proving statement 1.

In the conditions of statement 2, it is $(\lambda_i l_r/V_r) \ll 1$ for $1 \leq i \leq m$ and $(\lambda_i l_r/V_r) = O(1)$ for $m + 1 \leq i \leq n$, and, accordingly, the contribution of the first m elements in the diagonal matrix $\mathbf{B}_1$ is negligible.

If the n dimensional vector $\mathbf{B}_1$ is considered as sum of an m dimensional vector $\mathbf{F}$ and an $(n-m)$ dimensional vector $\mathbf{G}$ defined by

$$F_\alpha = B_{1\alpha} \qquad G_\gamma = B_{1\gamma} \qquad \begin{aligned} 1 \leq \alpha \leq m \\ m+1 \leq \gamma \leq n \end{aligned}$$

and if one introduces the following partitions of the matrices $\mathbf{N}$ and $\mathbf{N}^{-1}$:

$$\begin{aligned} M_{\alpha i} &= N_{\alpha i} \qquad & Z_{\gamma i} &= N_{\gamma i} \qquad & 1 \leq i &\leq n \\ & & & & 1 \leq \alpha &\leq m \\ R_{i\alpha} &= N_{i\alpha}^{-1} \qquad & \Omega_{i\gamma} &= N_{i\gamma}^{-1} \qquad & m+1 \leq \gamma &\leq n \end{aligned}$$

then Eq. [4] can be split in the following two equations:

$$\begin{aligned} \mathbf{F} &= \mathbf{M}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) \\ \mathbf{G} &= \mathbf{Z}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) - \boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \end{aligned}$$

and the expression for p_1 becomes, successively, through Eq. [5]

$$\begin{aligned} p_1 + (a_{e0}^2/v_0^2)v_1 &= -\boldsymbol{\varphi}\cdot\mathbf{N}^{-1}\cdot\mathbf{B}_1 = -\boldsymbol{\varphi}\cdot[\mathbf{R}\cdot\mathbf{F} + \boldsymbol{\Omega}\cdot\mathbf{G}] \\ &= -\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) + \boldsymbol{\varphi}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \\ &= -\nu_v\nabla\cdot\mathbf{V}_1 + \boldsymbol{\varphi}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \end{aligned}$$

thus proving (statement 2) that the subject motion is equivalent to the motion of a viscous mixture in which only $(n - m)$ reactions take place. The basic unknowns associated with the reactions are, in general, the $(n - m)$ linear combination of the affinities A_i given by

$$G_\gamma = \sum_{i=1}^{n} N_{\gamma i}^{-1}A_i \qquad m+1 \leq \gamma \leq n$$

wherein some of the components $N_{\gamma i}^{-1}$ may turn out to be negligible, for they are functions of the λ_i's, and, in the present case, it is $(\lambda_\alpha/\lambda_\gamma) \ll 1$. In particular, if the reactions are uncoupled (i.e., $L_{ij} = 0$ for $i \neq j$), the G_γ's reduce to the $(n - m)$ affinities A_γ's.

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Stagnation Region in Rarefied, High Mach Number Flow[1]

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THE departure at low Reynolds number from the concept of a thin boundary layer flow has been studied in Refs. 1–6 and in other works. In Refs. 2–5, the linear viscosity-temperature relation greatly simplifies the analyses, but the resultant stagnation point heat transfer rates obtained for the vorticity-interaction regime appear to differ considerably from that of Hayes and Probstein (1),[4] who adopt a Sutherland viscosity law. This difference suggests that the linear viscosity temperature relation may not be adequate for low Reynolds number studies. However, in this note it will be demonstrated that the linear representation of the viscosity law, with an appropriate reference temperature, is adequate for most purposes in both the high and the low Reynolds number regimes.[5] Also included is a comparison of the results of viscous shock-layer theory (5,6) with the recent heat transfer measurements of Ferri and Zakkay (7).

Viscosity-Temperature Relation

In Ref. 5, the viscosity-temperature relation is represented by the linear law

$$\mu = \mu_*(T/T_*) \qquad [1]$$

where μ_* is the viscosity evaluated at the reference temperature T_*. This latter quantity is chosen as

$$T_* = (T_s + T_w)/2 \qquad [2]$$

for the stagnation region, where T_s and T_w are the temperatures immediately behind the shock and at the body surface, respectively.

Simplifications of this type may not always lead to accurate solutions, especially for the details of the flow field near a cold surface. However, Eckert (8) and others have shown that this linear representation is generally adequate for predicting the skin friction and heat transfer characteristics of compressible boundary layers if the reference temperature is chosen at an appropriate level. When specialized to the stagnation point boundary layer, Eckert's reference temperature is $T_* = (T_0 + T_w)/2$, where T_0 is the stagnation temperature. The reference temperature is essentially that given by Eq. [2], since for this case $T_s \approx T_0$.

In the following discussion, the particular case of $\mu \propto T^\omega$ is compared with the case of the linear viscosity-temperature law, Eq. [1]. The exponent ω is taken to be 0.65, which compares reasonably well with the Sutherland law in the range of $T = 500°$ to $3000°$K for air. Numerical solutions of the viscous shock-layer equations (5) for the axisymmetric stagnation region (based on a model of ideal gas with constant specific heats) have been obtained using the nonlinear viscosity law for $Pr = 0.71$, $\epsilon = p/2\rho h = (\gamma - 1)/2\gamma = 0.10$ and $T_w/T_0 = 0.10$ over a wide range of the Reynolds number

$$Re_b = \rho_\infty U_\infty a/\mu_0 \qquad [3]$$

Received by ARS August 31, 1962; revision received October 22, 1962.

[1] This research was supported by the Aeronautical Research Laboratory of the Office of Aerospace Research, U. S. Air Force, under Contract AF 33(616)-7812, and by the U. S. Navy through the Office of Naval Research (Mechanics Branch), Contract Nonr 2653(00).

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[4] Numbers in parentheses indicate References at end of paper.

[5] The reason for the difference between the results of Ref. 1 and others is discussed in Ref. 6.

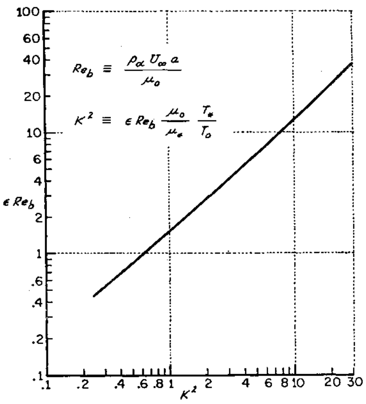

Fig. 1 Conversion of ϵRe_b to K^2 for stagnation regions assuming Pr = 0.71, $\mu_\infty \propto T^{2/3}$, $T_w/T_0 \rightarrow 0$, and ϵ = 0.10 to 0.15

Here Pr stands for the Prandtl number, p the pressure, ρ the density, h the enthalpy, ρ_∞ and U_∞ are the freestream values of ρ and u, a is the nose radius, and μ_0 is the viscosity at the stagnation temperature. These results are compared with corresponding solutions (5) obtained using the linear viscosity law, Eq. [1], for Pr = 0.71, ϵ = 0.10, and $T_w/T_0 \rightarrow 0$. In the calculations using the nonlinear viscosity law, a small but finite ratio T_w/T_0 = 0.10 is used instead of $T_w/T_0 \rightarrow 0$ in order to avoid singularities in the differential equations.

To compare the solution based on the linear viscosity law given in Ref. 5 with that based on the nonlinear law, one must relate Re_b to the rarefaction parameter K^2. According to Ref. 5,

$$K^2 \equiv \epsilon_\infty \frac{\rho_\infty U_\infty a}{\mu_*} \frac{T_*}{T_0} = \epsilon_\infty Re_b \left(\frac{T_s + T_w}{2T_0}\right)^{1-\omega} \qquad [4]$$

Here, the temperature immediately behind the shock T_s is to be supplied by the solution based on the linear viscosity relation. For the present consideration, $(T_s + T_w)/T_0 \approx T_s/T_0$. The ratio T_s/T_0 for the axisymmetric case has been given in Fig. 3 of Ref. 5 as a function of K^2. Thus, for each value of K^2, one can determine the corresponding value of ϵRe_b from Eq. [4]. The relation is presented in Fig. 1 for $\omega = \frac{2}{3}$, $T_w/T_0 \rightarrow 0$, and Pr = 0.71.[6]

The results of these calculations are presented and compared in Fig. 2. Included in this figure are surface heat transfer rate, skin friction, enthalpy immediately behind the shock, and the shock standoff distance. These quantities are presented as functions of K^2.

Except for the standoff distance, the small differences in T_w/T_0 (0.10 and 0) are not expected to affect appreciably the quantities being compared. In order to compare the standoff distances (the distance between the body and the inner edge of the shock-transition zone) at the same wall-to-stagnation temperature ratio, the quantity $\Delta/\epsilon a$ pertaining to the linear viscosity-temperature relation has been computed from the approximate, analytical solution of Ref. 5, using T_w/T_0 = 0.10. The agreement between the analyses based on the linear and the nonlinear viscosity laws, as revealed by the comparison presented in Fig. 2, is excellent indeed. In both the regime where the Rankine-Hugoniot relations hold and the regime where the transport effects behind the shock are significant, the difference between the two cases is at most 4% for the four quantities shown.

Comparison of Theory and Experiment

As a check on the validity of the shock-layer theory, the authors will provide a comparison of theory and experiment for the stagnation-point heat transfer in the axisymmetric case. A similar comparison has been made previously in Ref. 5. There, most of the experimental data used lie in the range of $K^2 \gtrsim 10$ and generally are consistent with the theory. Recently, Ferri and Zakkay have obtained measurements in a range of lower Reynolds number, extending the K^2 value to as low as unity (7). The experiments in Ref. 7 were performed at a stagnation temperature of 900° to 1300°K, with a test-flow Mach number of 5.23 to 5.74. These data together with some of those obtained earlier in Ref. 9, are shown in Fig. 3 as $(Q - Q_{BL})/Q_{BL}$ vs Re_F. Here, Q is the heat transfer rate at the wall, and the subscript BL stands for the boundary-layer solution. The Reynolds number Re_F is defined as

$$Re_F \equiv \rho_t(H_\infty^{1/2}\, a)/\mu_t \qquad [5]$$

where H_∞ is the total enthalpy in the freestream and the subscript t refers to the condition at the stagnation point as provided by the inviscid theory. The parameters Re_F and K^2 are related through $Re_F \approx Re_b/(2)^{1/2}\epsilon$.

Allowing vibrational equilibrium, the value of ϵ is about 0.13 behind the shock (presumably somewhat higher within the shock layer). Although the surface temperature of the test models T_w has not been provided explicitly in Refs. 7 and 9, an estimate shows that the ratio T_w/T_0 is about $\frac{1}{3}$ or smaller.

[6] The relation between the parameters ϵRe_b and K^2 also has been determined for the case of an unyawed cylinder. The difference between the two cases is found to be very small.

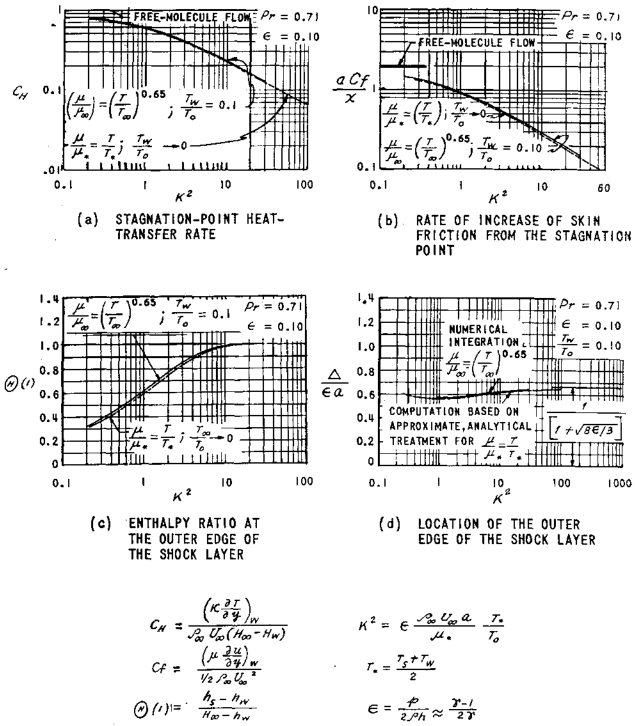

Fig. 2 Comparisons of theoretical results of axisymmetric stagnation region based on an exact nonlinear viscosity law and on a linear law using a reference temperature at various degrees of rarefaction

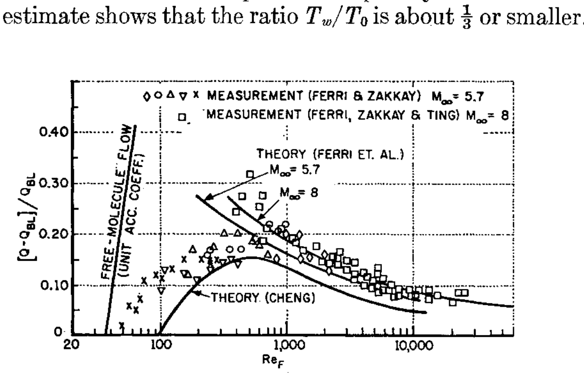

Fig. 3 A comparison of theory and experiment for stagnation point heat transfer rate (note that Cheng's result is calculated for $M_\infty \rightarrow \infty$, $T_w/T_0 \rightarrow 0$, Pr = 0.71, and ϵ = 0.13)

The theoretical curve of Cheng's shock-layer theory presented in Fig. 3 has been deduced from the heat transfer coefficient C_H of Ref. 5 (refer to Fig. 6 of Ref. 5) for $Pr = 0.71$, $\epsilon = 0.13$, and $T_w/T_0 \to 0$. In converting K^2 to Re_F, Fig. 1 has been used. The small differences in T_w/T_0 and ϵ between the experimental and theoretical data are not believed to affect significantly the quantities presented in the correlation. The data presented for comparison encompass both the regime where vorticity interaction dominates ($Re_F \gtrsim 500$) and the regime where the transport effects immediately behind the shock are important, $Re_F \gtrsim 500$. Although the experimental data appear to rise somewhat above Cheng's values, especially in the higher Reynolds number range, the comparison in Fig. 3 indicates a general agreement between experiment and the shock-layer theory to a degree consistent with the approximation of the theory. Also included in Fig. 3 are two theoretical curves of Ferri et al. (4,7).

A somewhat similar comparison has been given recently by Ferri, Zakkay, and Ting (9), covering mainly the higher Reynolds number regime ($Re_F \gtrsim 500$). There, the agreement between the heat transfer measurement and the prediction based on the shock-layer theory of Ref. 5 seems to be even better.[7]

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[7] The apparent improvement in the agreement between experiment and the shock-layer theory reported in Ref. 9 results primarily from using the Fay and Riddell formula for Q_{BL} instead of the value obtained from the shock-layer analysis. Since the theoretical values of heat transfer rate are given in Ref. 5 as a leading approximation for small ϵ, the quantity $(Q - Q_{BL})/Q_{BL}$ based on this theory is valid (to the leading approximation) only if Q_{BL} is also taken from the leading approximation for small ϵ. Using Fay and Riddell's formula for Q_{BL} would result in misleading values of $(Q - Q_{BL})/Q_{BL}$.

Optimum Thrust Programming of Electrically Powered Rocket Vehicles for Earth Escape

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Nomenclature

a = thrust per unit mass of vehicle
m = mass of propellant expended at time t
M_0 = initial mass of vehicle plus propellant
P = jet power
r = radius measured from center of earth
s = distance measured along flight path
U = gravitational potential, $U = \mu/r$
V = velocity of vehicle
α = $(d^2U/ds^2)^{1/2}$
μ = constant of gravitational field
ϕ = flight path angle measured from local horizontal

Subscripts

0 = initial value at $t = 0$
T = final value at $t = T$

Optimum thrust programming of electrically powered rockets under the conditions of constant jet power and tangential thrust was discussed in Ref. 1. Assuming the gradient along the flight path of the gravitational force per unit mass to be constant, thrust programs were derived yielding minimum propellant utilization for the two particular cases of specified change in velocity with range arbitrary and specified range with change of velocity arbitrary. A practical problem that is actually a combination of these two cases is that of escape from a satellite orbit with a minimum expenditure of propellant. Here, neither the change in velocity nor the range is specified, but a functional relationship exists between these two quantities.

If the nomenclature of Ref. 1 is used, the equations of motion are

$$\dot{V} = a + (dU/ds) \quad [1]$$

$$\dot{s} = V \quad [2]$$

where U, the gravitational potential, is assumed to be a function of s only. The propellant mass is uniquely determined by the function

$$\psi = 2P/(M_0 - m) \quad [3]$$

where

$$\dot{\psi} = a^2 \quad [4]$$

At the specified time of burnout, T, the relationship between the velocity and the distance is given by

$$(V_T^2/2) - U_T = 0 \quad [5]$$

Upon applying the formal methods of the calculus of variations to the problem of determining $a(t)$ for which ψ_T is stationary under the restrictions imposed by Eqs. [1, 2, and 4], the differential equation

$$\ddot{a} = (d^2U/ds^2)a \quad [6]$$

is obtained, together with the condition

$$-a_T\delta V_T + \dot{a}_T\delta s_T = 0 \quad [7]$$

at time T (1).[2] From Eq. [5]

$$V_T\delta V_T - (dU/ds)_T\delta s_T = 0 \quad [8]$$

The combination of Eqs. [7] and [8] yields

$$\frac{\dot{a}_T}{a_T} = \frac{(dU/ds)_T}{V_T} \quad [9]$$

In order to obtain analytical solutions of Eq. [6], it is convenient to assume that d^2U/ds^2, the gradient of the tangential component of gravitational force per unit mass, is constant. The solution of Eq. [6] for which $a = a_0$ at $t = 0$ and Eq. [9] is satisfied at $t = T$ is then

$$\frac{a}{a_0} = \frac{\cosh\alpha T\left(1 - \frac{t}{T}\right) - K\sinh\alpha T\left(1 - \frac{t}{T}\right)}{\cosh\alpha T - K\sinh\alpha T} \quad [10]$$

Received by ARS September 18, 1962.

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[2] Numbers in parentheses indicate References at end of paper.